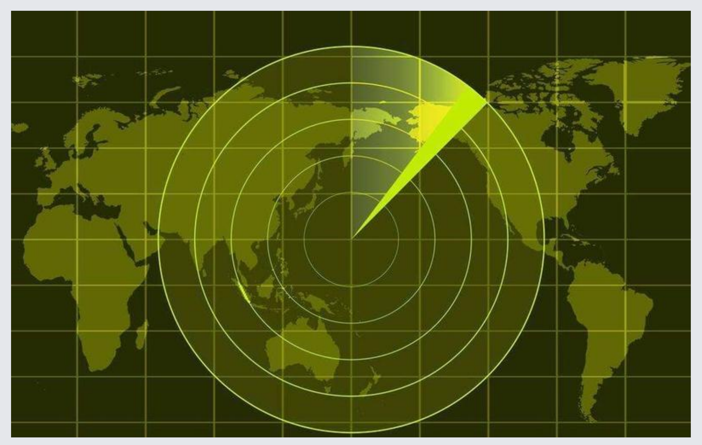
Complex-valued Neurons Can Learn More but Slower than Real-valued Neurons via Gradient Descent

Jin-Hui Wu, Shao-Qun Zhang, Yuan Jiang, and Zhi-Hua Zhou
{ wujh, zhangsq, jiangy, zhouzh }@lamda.nju.edu.cn

Motivation

- Complex-valued neural networks (CVNNs) can outperform real-valued neural networks (RVNNs) in some signal processing tasks.
- But CVNNs cannot always outperform RVNNs.



Radar Signals

Audio Signals

- Two important questions:
  - When CVNNs outperform RVNNs via gradient descent (GD)?
  - Can we learn everything with CVNNs without paying additional price?
- We answer from the aspect of neuron learning, i.e., learning a single neuron using another neuron.
  - It is a special case of neural network learning.
  - Its analysis is tractable.
  - It is sufficient to tell us the difference between RVNNs and CVNNs.

Formulation

- In this paper, neuron learning minimizes the expected square loss via GD

$$L(\boldsymbol{w}, \psi_w) = \mathbb{E}_{\boldsymbol{x}\sim\mathcal{D}}\left[\left(\sigma_{\psi_w}(\boldsymbol{w}^\top\overline{\boldsymbol{x}}) - \sigma_{\psi_v}(\boldsymbol{v}^\top\overline{\boldsymbol{x}})\right)^2\right].$$

- Here:
  - $\mathcal{D} = \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ is the distribution of input $\boldsymbol{x} \in \mathbb{C}^d$.
  - $(\psi_v, \boldsymbol{v})$ is the fixed target neuron.
  - $(\psi_w, \boldsymbol{w})$ is the learnable neuron with random initialization.
  - σ_ψ is the symmetric zReLU activation function

$$\sigma_\psi(z) = \begin{cases} \mathrm{Re}(z), & \theta_z \in [-\psi, \psi], \\ 0, & \text{otherwise.} \end{cases}$$

  - For a complex-valued neuron (CVN), both ψ_w and $\boldsymbol{w}$ are learnable.
  - For a real-valued neuron (RVN), ψ_w is fixed as $\frac{\pi}{2}$, only $\boldsymbol{w}$ is learnable, and σ_ψ degenerates to the ReLU activation function.

CVNs Can Learn More than RVNs

Two positive learning results for CVNs.

Theorem 1 (informal)**.** Let $d = 1$, and L_{cr} is the expected loss of learning an RVN using a CVN via GD. Under random initialization, if the step size of GD satisfies $\eta_t = \eta \in (0, 1/(12\pi))$, then we have

$$\Pr[L_{\mathrm{cr}} = O(t^{-3})] > 0.$$

A CVN learns an RVN at rate $O(t^{-3})$.

Theorem 2 (informal)**.** Let $d = 1$, and L_{cc} is the expected loss of learning a CVN using a CVN via GD. Under random initialization, if the step size of GD satisfies $\eta_t = \min\{c_1, c_2/t\}$ with $c_1 \le 1/3000$ and $c_2 \ge 20$, then we have

$$\Pr[L_{\mathrm{cc}} = O(t^{-1})] > 0.$$

A CVN learns a CVN at rate $O(t^{-1})$.

One negative learning results for RVNs.

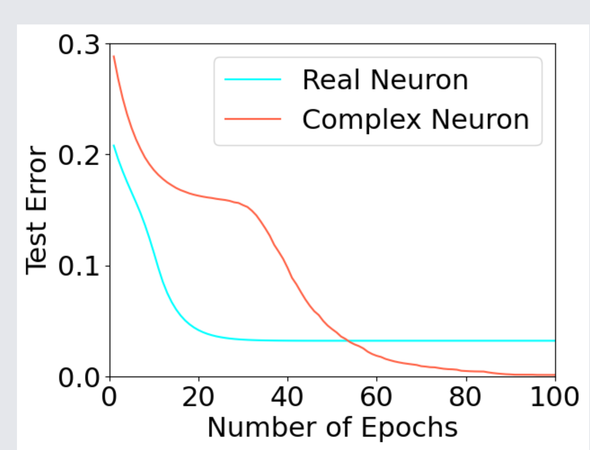
- The phase ψ of an RVN is fixed as $\pi/2$.
- An RVN has less learnable parameters than a CVN.
- The negative result considers learning a CVN using a two-layer RVNN for fairness.

Theorem 4 (informal)**.** Let $d = 1$, and L_{rc} is the expected loss of learning a CVN using a two-layer RVNN with n hidden neurons. If the CVN is non-degenerate, i.e., $\psi_v \notin \{0, \pi/2\}$ and $\boldsymbol{v} \neq 0$, then we have

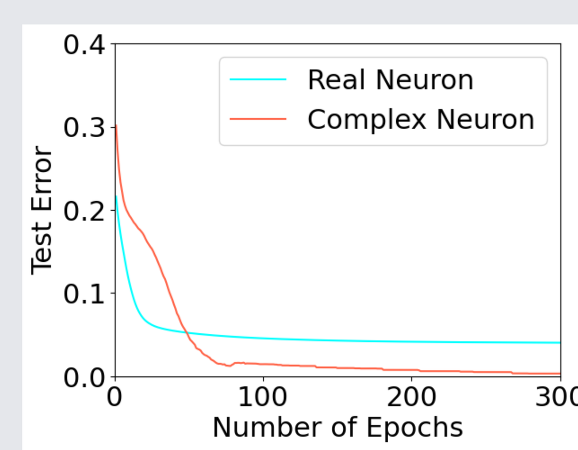
$$L_{\mathrm{rc}} \ge \frac{\|\boldsymbol{v}\|^2 \min\{2\psi, \pi - 2\psi\}^3}{24\pi(n+2)^2} > 0.$$

RVNNs with fixed width cannot learn a non-degenerate CVN.

Simulation experiments.



$d = 1$ and no bias term. (theoretical setting)



$d = 5$ and with bias term. (general setting)

CVNs Learn Slower than RVNs

Lemma 5 [Yehudai and Shamir, 2020] (informal)**.** Let L_{rr} be the expected loss of learning an RVN using an RVN via GD. Under random initialization and suitable step size, we have

$$\Pr[L_{\mathrm{rr}} = O(\mathrm{e}^{-ct})] > 0.$$

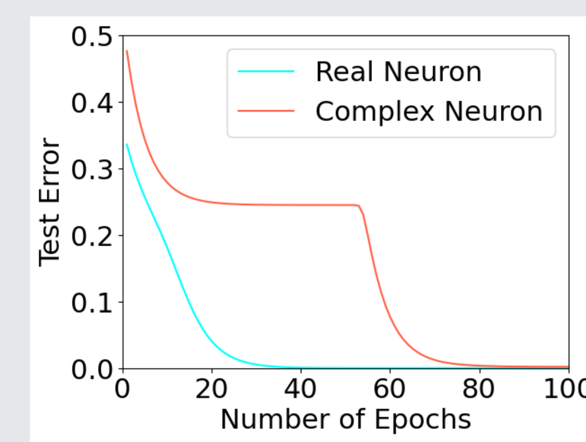
An RVN learns an RVN with exponentially small loss.

Theorem 6 (informal)**.** Let $d = 1$, and L_{cr} is the expected loss of learning an RVN using a CVN via GD. If the initialization is around the global minimum, and the step size of GD satisfies $\eta_t = \eta \in (0, 1/(12\pi))$, then we have

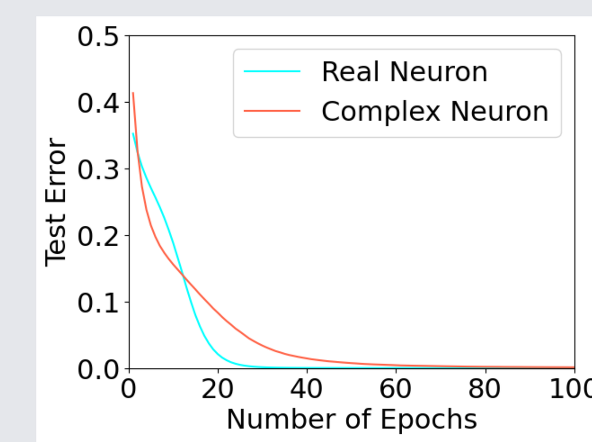
$$L_{cr} = \Omega(t^{-3}).$$

A CVN learns an RVN with polynomially large loss.

Simulation experiments.



$d = 1$ and no bias term. (theoretical setting)



$d = 5$ and with bias term. (general setting)

Summary

- CVNs can learn more than RVNs
  - A CVN learns an RVN at rate $O(t^{-3})$.
  - A CVN learns a CVN at rate $O(t^{-1})$.
  - RVNNs with fixed width cannot learn a non-degenerate CVN.
- CVNs learn slower than RVNs.
  - An RVN learns an RVN at rate $O(\mathrm{e}^{-ct})$.
  - A CVN learns an RVN at rate $\Omega(t^{-3})$.

Target	RVN	CVN
RVN	$O(e^{-ct})$	$\Theta(t^{-3})$
CVN	Cannot Learn	$O(t^{-1})$

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